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Totally Symmetric Self-Complementary Plane Partitions and Quantum Knizhnik-Zamolodchikov equation: a conjecture

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We present a new conjecture relating the minimal polynomial solution of the level-one $U_q(\mathfrak{sl}(2))$ quantum Knizhnik-Zamolodchikov equation for generic values of q in the link pattern basis and some q-enumeration of Totally Symmetric Self-Complementary Plane Partitions.

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1. Introduction

Statistical physics have always nurtured a particular relationship with enumerative combinatorics. A remarkable manifestation of this fact is the recent conjecture by Razumov and Stroganov (RS) [1], relating the solution of the O(1) dense loop model on a semi-infinite cylinder of square lattice with perimeter $2n$ to the refined enumeration of $n \times n$ Alternating Sign Matrices (ASM) in the (bijectively equivalent) form of configurations of the Fully-Packed Loop (FPL) model on an $n \times n$ square grid.

An actual proof [2] of a weaker *sum rule* version of the RS conjecture [3] was actually constructed by making full use of the integrability of the O(1) model. The latter allowed to completely determine the vector of (renormalized) probabilities for the configurations of an inhomogeneous version of the model on the cylinder to connect boundary points according to given connectivity patterns, and the sum rule followed. An alternative proof was found in [4], in the course of studying polynomial representations of the affine Temperley-Lieb algebra. The latter provide a deformation of the original model, involving a parameter q. This was recast in [5] as the problem of finding non-trivial *minimal polynomial* solutions of the quantum Knikhnik-Zamolodchikov (qKZ) equation for the level one quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ [6], the particular "RS" value of q being $q = -e^{i\pi/3}$. It was also noted in [5] that another particular value of q, $q = -1$, actually contains lots of combinatorial wonders. Indeed, when taking a suitable $q \to -1$ limit of the homogeneous version of the above probabilities, one ends up with non-negative integers which were shown to count the degrees of the components of the variety $M^2 = 0$, where M is a complex strictly upper triangular matrix, thus providing another RS-type theorem, in relation with algebraic geometry.

In the homogeneous case but with q generic, one gets a remarkable sum rule for the abovementioned probabilities, in the form of polynomials of the variable

$$\tau = -q - q^{-1} \tag{1.1}$$

with apparently only *non-negative integer* coefficients. The aim of this note is to provide a conjectural answer for what these coefficients actually count.

In the long story of the ASM conjecture (see Bressoud's book [7] for a complete saga and references), many puzzles remain open, one of which is the still elusive relation between ASM of size $n \times n$ and the Totally Symmetric Self-Complementary Plane Partitions (TSSCPP) of size $2n$, or alternatively the rhombus tilings of a regular hexagon of triangular

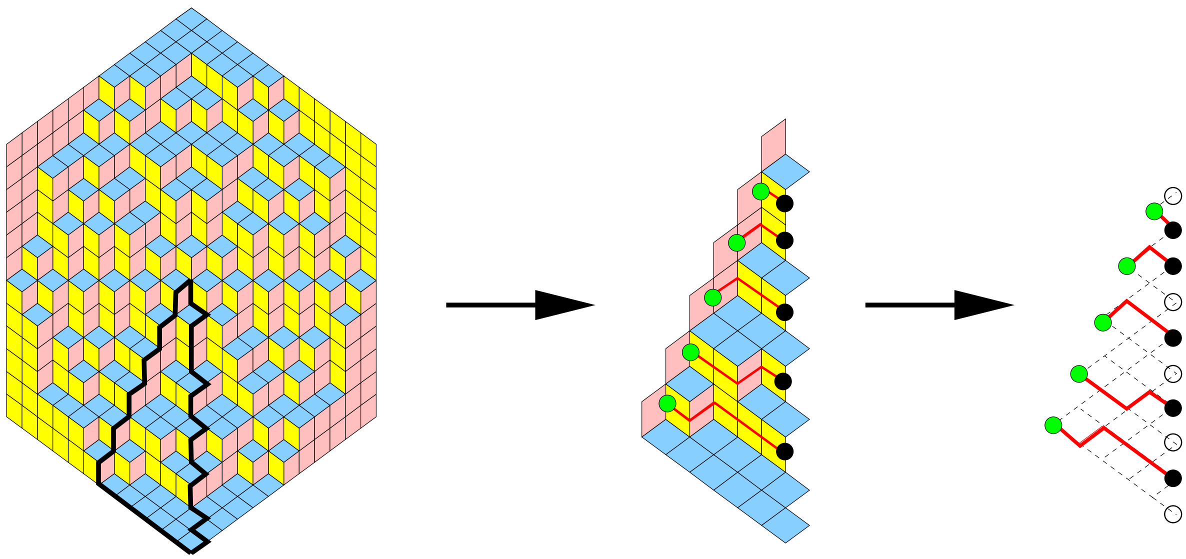
lattice, with side of length $2n$, with all symmetries of the hexagon and also the condition that is be identical to its complement when interpreted as the view in perspective of a piling up of unit cubes within a cube of size $2n$. Despite many efforts, no natural bijection has yet been found between ASM and TSSCPP. It is however rather simple to enumerate TSSCPP, via a bijection to a set of Non-Intersecting Lattice Paths (NILP), which in turn are counted by determinant formulas.

In this note, we present the following conjecture:

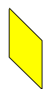
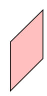
The weighted enumeration of TSSCPP of size $2n$, in the form of lattice paths with two types of steps, and with a weight $\tau = -q - q^{-1}$ per step of the first type produces the generic q sum rule for the above minimal polynomial solution of the level-one $U_q(\mathfrak{sl}(2))$ qKZ equation.

If true, this conjecture provides a new relation between TSSCPP and ASM, although not yet one-to-one, but rather many-to-many. It also suggests that the RS conjecture may be extended to one that allows to interpret the abovementioned probabilities in the homogeneous case as weighted counting functions for restricted classes of TSSCPP, as each of these quantities also appears to be a polynomial of τ with non-negative integer coefficients. This ultimate refinement would pave the way to a natural TSSCPP-ASM bijection, yet to be found.

The paper is organized as follows. In Section 2, we review briefly the enumeration of TSSCPP by reinterpreting them as NILP. The latter is refined by introducing a weight τ per vertical step, leading to polynomials $P_{2n}(\tau)$, also expressed via a simple Pfaffian formula. In Section 3, we review briefly the minimal polynomial solution of the level-one $U_q(\mathfrak{sl}(2))$ qKZ equation in the link pattern basis, and concentrate on the sum of components in the homogeneous case, leading to polynomials $\Pi_{2n}(\tau)$. This leads to the main conjecture of the paper, namely that $P_{2n}(\tau) = \Pi_{2n}(\tau)$ (Sect. 4.1) which we check at $\tau = 2$ and then at generic τ by numerically solving the qKZ equation by means of a new algorithm, based on the decomposition of the solution as a sum of products of monomials of a particular type (Sect. 4.2). The main consequences of the conjecture as well as additional remarks are gathered in the concluding Section 5.



TSSCPP **NILP**

Fig. 1: A typical TSSCPP of size $2n = 12$. The area delimited by a black broken line is a fundamental domain for all symmetries of the partition (1/12th of the hexagon). This domain is expressed as a configuration of NILP, by following the sequences of rhombic tiles  and .

2. TSSCPP

The counting of TSSCPP may be best performed by using their formulation as NILP. To make a long story short, we simply give a pictorial representation of the TSSCPP-NILP correspondence in Fig.1. Starting from a TSSCPP in the from of a totally symmetric self-complementary rhombus tiling of a regular hexagon of triangular lattice with edge of length $2n$, we concentrate on a fundamental domain made of 1/12th of the hexagon, and note that its tiling configuration is entirely determined by the configurations of $n-1$ sequences of rhombi of two of the three types used (see middle picture in Fig.1). The latter form NILP, which are represented on the right of Fig.1.

Upon trivial deformations of the underlying triangular lattice and harmless rotations and reflexions, the problem boils down to that of counting of NILP on the square lattice, starting from the points $(i, -i)$, $i = 1, 2, \ldots, n-1$ and ending at positive integer points on the x axis, of the form $(r_i, 0)$, $i = 1, 2, \ldots, n-1$, and making "vertical" steps $(0, 1)$ or "diagonal" ones $(1, 1)$ only (see Fig.2 for illustration). It is clear that the i-th path, starting from $(i, -i)$ must end at $(r_i, 0)$, as the paths do not intersect each other. Moreover, we must have $r_i \leq 2i$ for all i, and the sequence r_i is strictly increasing.

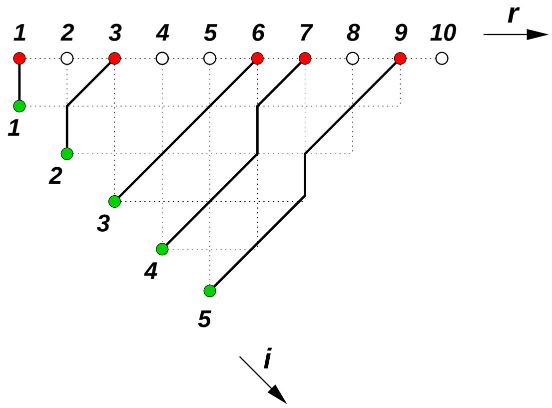


Fig. 2: A sample NILP for $n = 6$. The paths make only up $(0,1)$ and right-diagonal $(1,1)$ steps on the underlying square lattice, and start from the points $(i,-i)$ (green dots) and end up on the x axis (red dots), at points $(r_i,0)$, for $i = 1,2,\ldots 5$.

The counting of paths with prescribed ends r_i, $i = 1,2,\ldots,n-1$ is readily performed by use of the Lindström-Gessel-Viennot (LGV) formula [8], expressing the total number of NILP configurations as a determinant:

$$P_{2n}(r_1,r_2,\ldots,r_{n-1}) = \det_{1\leq i,j\leq n-1} \mathcal{P}(i,r_j) \tag{2.1}$$

where $\mathcal{P}(i,r)$ denotes the number of paths from $(i,-i)$ to $(r,0)$. This latter number is nothing but

$$\mathcal{P}(i,r) = \binom{i}{r-i} \tag{2.2}$$

as we simply have to choose $r-i$ diagonal steps among a total of i. Note that the number of vertical steps is therefore $i-(r-i) = 2i-r$. Hence, if we wish to obtain the weighted enumeration of TSSCPP with prescribed endpoints r_i and with a weight τ per vertical step, we simply have to multiply $\mathcal{P}(i,r)$ by τ^{2i-r} in (2.1). This yields the weighted enumeration of TSSCPP with prescribed endpoints:

$$P_{2n}(r_1,r_2,\ldots,r_{n-1}|\tau) = \det_{1\leq i,j\leq n-1}\left(\binom{i}{r-i}\tau^{2i-r}\right) \tag{2.3}$$

and the total weighted sum of TSSCPP of size $2n$ reads then

$$P_{2n}(\tau) = \sum_{\substack{1\leq r_1<r_2<\cdots<r_{n-1}\\ r_i\leq 2i}} P_{2n}(r_1,r_2,\ldots,r_{n-1}|\tau) \tag{2.4}$$

It is a polynomial in τ of degree $n(n-1)/2$, the total number of steps of each configuration.

For completeness we list below the first few values of this polynomial.

$$\begin{aligned}
P_2(\tau) &= 1 \\
P_4(\tau) &= 1+\tau \\
P_6(\tau) &= 1+3\tau+2\tau^2+\tau^3 \\
P_8(\tau) &= 1+6\tau+11\tau^2+12\tau^3+8\tau^4+3\tau^5+\tau^6 \\
P_{10}(\tau) &= 1+10\tau+35\tau^2+70\tau^3+98\tau^4+91\tau^5+69\tau^6+35\tau^7+15\tau^8 \\
&\quad +4\tau^9+\tau^{10} \\
P_{12}(\tau) &= 1+15\tau+85\tau^2+275\tau^3+628\tau^4+1037\tau^5+1346\tau^6+1379\tau^7+1144\tau^8 \\
&\quad +783\tau^9+435\tau^{10}+204\tau^{11}+74\tau^{12}+24\tau^{13}+5\tau^{14}+\tau^{15}
\end{aligned} \tag{2.5}$$

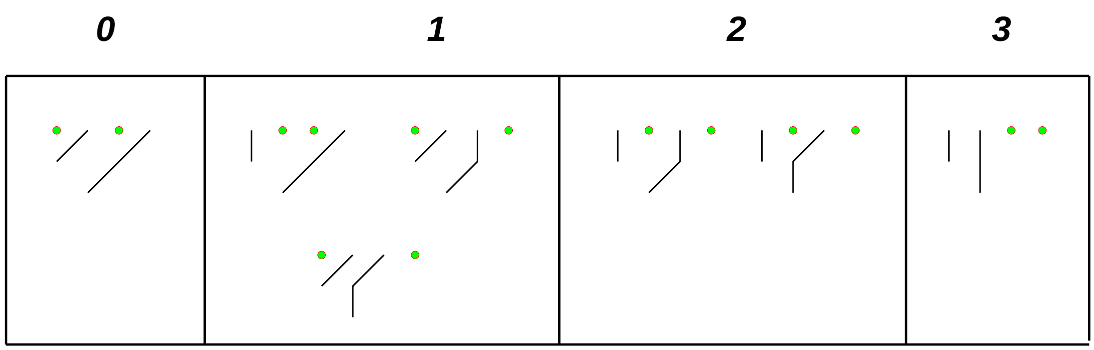


Fig. 3: The case of size $2n = 6$. We have represented the TSSCPP with fixed numbers of vertical steps, ranging from 0 to 3 (indicated in the top row), leading to the polynomial $P_6(\tau) = 1 + 3\tau + 2\tau^2 + \tau^3$.

For illustration, we have represented in Figs. 3 and 4 the TSSCPP of size $2n = 6$ and 8 respectively, arranged according to their numbers of vertical steps.

At $\tau = 1$, $P_{2n}(1)$ reproduce the celebrated TSSCPP numbers usually denoted by $N_{10}(2n, 2n, 2n)$, also equal to the total numbers of ASM of size $n \times n$: $N_{10}(2n, 2n, 2n) = P_{2n}(1) = A_n = \prod_{0\le j\le n-1} \frac{(3j+1)!}{(j+n)!}$, with values $1, 2, 7, 42, 429, \ldots$ for $n = 1, 2, 3, 4, 5, \ldots$

At $\tau = -1$, we get an interesting sequence $1, 0, -1, 0, 9, 0, -646, \ldots$ for $n = 1, 2, 3, 4, 5, 6, 7 \ldots$, which we have been able to (conjecturally) relate to the so-called refined ASM numbers $A_{n,k}$ (see [7] and references therein), counting ASM of size $n \times n$ with the unique 1 in the first row at position k, with values

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} \tag{2.6}$$

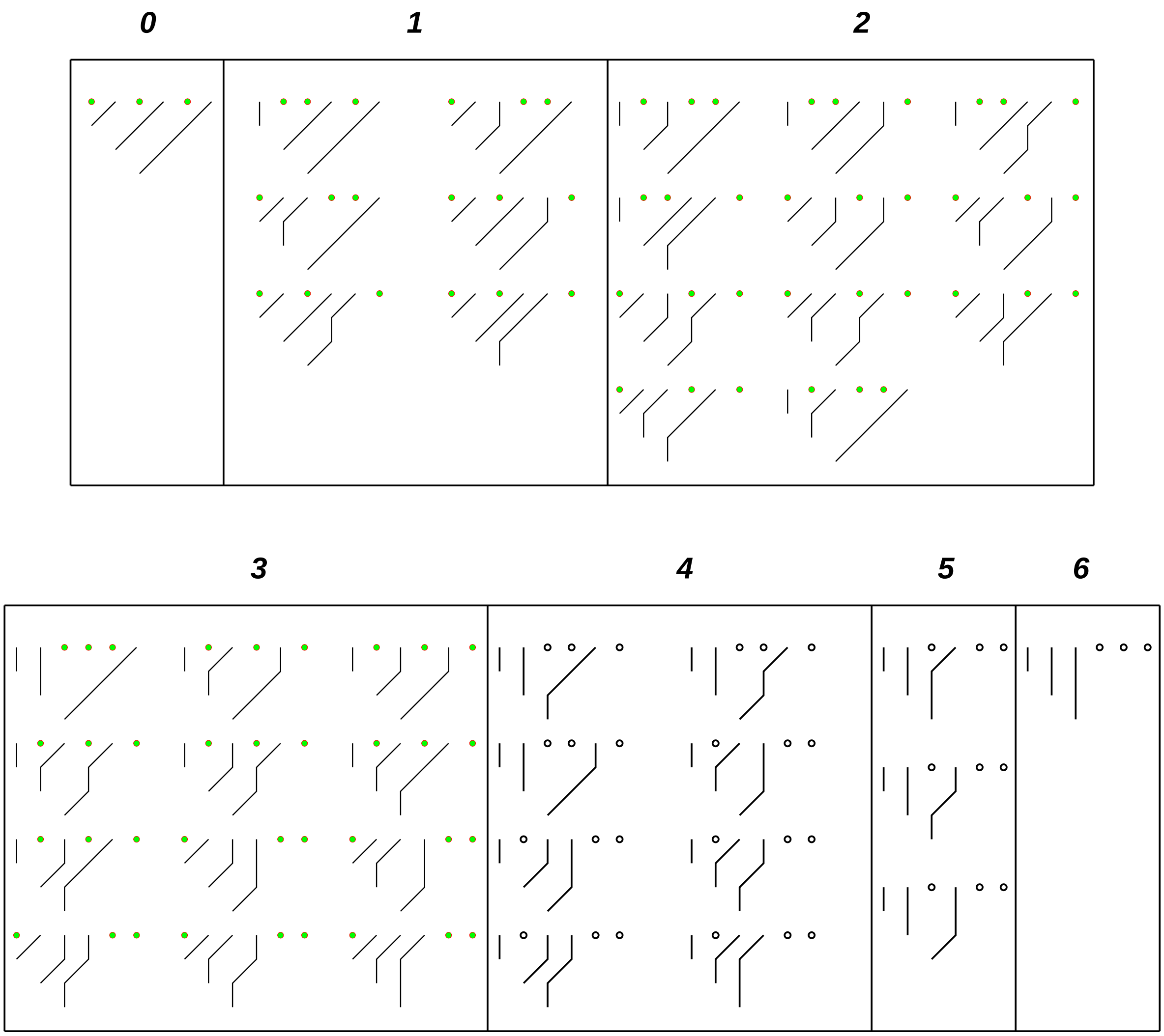


Fig. 4: The case of size $2n = 8$. We have represented the TSSCPP with fixed numbers of vertical steps, ranging from 0 to 6 (indicated in the top row), leading to the polynomial $P_8(\tau) = 1 + 6\tau + 11\tau^2 + 12\tau^3 + 8\tau^4 + 3\tau^5 + \tau^6$.

The observed relation reads:

$$P_{2n}(-1) = \sum_{k=1}^{n} (-1)^{\frac{n+1}{2}-k} A_{n,k} \tag{2.7}$$

Note that this vanishes when n is even, and that moreover when n is odd, the result is a perfect square up to a sign, which we have identified as:

$$P_{4n+2}(-1) = (-1)^n A_V(2n+1)^2 \tag{2.8}$$

where $A_V(2n+1)$ is the number of Vertically Symmetric ASM (VSASM), with value [9]:

$$A_V(2n+1) = \prod_{j=0}^{n-1} \frac{(6j+4)!(2j+2)!}{(4j+4)!(4j+2)!} \tag{2.9}$$

We have been able to check the relation (2.8) for n up to 6, i.e. for TSSCPP of size up to 26.

It has been known for quite some time that the TSSCPP are also enumerated via a Pfaffian expression [10], simply expressing in physical terms the fermionic character of non-intersecting lattice paths. We have found an easy way to generalize the Pfaffian expression to the present case. Let $\epsilon_n = (1-(-1)^n)/2$. Then introducing the quantity

$$H(i,j|\tau) = \sum_{i\leq r<s\leq 2j} \tau^{2i+2j-r-s} \left(\binom{i}{r-i}\binom{j}{s-j} - \binom{i}{s-i}\binom{j}{r-j} \right) \tag{2.10}$$

the polynomial $P_{2n}(\tau)$ is nothing but the Pfaffian:

$$P_{2n}(\tau) = \mathrm{Pf}\ \left(H(i,j|\tau)\right)_{\epsilon_n\leq i<j\leq n-1} \tag{2.11}$$

In particular, the above identification at $\tau=-1$ implies a new Pfaffian formula for the square of the number of VSASM, yet to be proved:

$$A_V(2n+1)^2 = (-1)^n \mathrm{Pf} \left(\sum_{i\leq r<s\leq 2j} (-1)^{r+s} \left(\binom{i}{r-i}\binom{j}{s-j} - \binom{i}{s-i}\binom{j}{r-j} \right) \right)_{1\leq i<j\leq 2n} \tag{2.12}$$

3. Minimal polynomial solution of the qKZ equation

The inhomogeneous O(1) dense loop model on a semi-infinite cylinder of square lattice of perimeter $2n$ was investigated in Ref.[2]. In particular, the probability $\mathcal{P}_\pi$ for a loop configuration to connect the $2n$ boundary points by pairs via non-intersecting links according to the link pattern π was computed. Here a link pattern is simply a chord diagram on a disk with $2n$ regularly spaced boundary points, numbered 1 to $2n$ counterclockwise, connected by pairs via n non-intersecting chords. There are $c_n = \frac{(2n)!}{n!(n+1)!}$ such link patterns. These correspond to the effective connections between the points on the boundary of the semi-infinite cylinder, via bulk configurations of non-intersecting links. The inhomogeneous character of the model is simply via Boltzmann weights that depend on the vertical position of the face configurations, via complex parameters $z_1, z_2, \ldots, z_{2n}$. In [2], the probabilities $\mathcal{P}_\pi$ were renormalized up to a global constant by a common overall

polynomial factor in such a way that they all be polynomials of the z's with no common factors. We denote by $\Psi_\pi \equiv \Psi_\pi(z_1, z_2, \dots z_{2n})$ these renormalized probabilities. The quantities Ψ_π were shown to obey the following system of equations

$$\begin{aligned} \tau_{i,i+1}\Psi(z_1,\dots,z_{2n}) &= \check{R}_{i,i+1}(z_i,z_{i+1})\,\Psi(z_1,\dots,z_{2n}), \quad i=1,2,\dots 2n-1 \\ \sigma\,\Psi(z_2,\dots,z_{2n},z_1) &= \Psi(z_1,\dots,z_{2n}) \end{aligned} \tag{3.1}$$

where we have used the following notations: $t_{i,i+1}$ for the elementary transposition $z_i \leftrightarrow z_{i+1}$; $\check{R}_{i,i+1}$ for the R-matrix operator acting linearly on the link pattern basis as

$$\check{R}_{i,i+1} = \frac{q^{-1}z_i - qz_{i+1}}{q^{-1}z_{i+1} - qz_i} Id + \frac{z_i - z_{i+1}}{q^{-1}z_{i+1} - qz_i} e_i \tag{3.2}$$

with $q = -e^{i\pi/3}$, and where e_i is the i-th generator of the Temperley-Lieb algebra acting on link patterns as follows: e_i creates a link between points i and $i+1$ while it glues the links connected to i and $i+1$ respectively (any loop created in the process may be safely erased, as the loop weight is $\tau = -q - q^{-1} = 1$); finally σ acts on link patterns as a counterclockwise rotation by one unit.

The system (3.1) may be alternatively written in components as

$$\begin{aligned} (q^{-1}z_{i+1} - qz_i)\partial_i \Psi_\pi &= \sum_{\substack{\pi' \neq \pi \\ e_i \pi' = \pi}} \Psi_{\pi'}, \quad \forall \pi \in \ \mathrm{Im}(e_i),\ 1 \le i \le 2n-1 \\ \Psi_\pi(z_2,\dots,z_{2n},z_1) &= \Psi_{\sigma(\pi)}(z_1,\dots,z_{2n}) \end{aligned} \tag{3.3}$$

where we have introduced the divided difference operator

$$\partial_i f(z_i, z_{i+1}) = \frac{f(z_{i+1},z_i) - f(z_i,z_{i+1})}{z_i - z_{i+1}} \tag{3.4}$$

which decreases by 1 the degree of the polynomial f. These equations were then solved by noticing first that all components Ψ_π may be obtained in a triangular way from the fundamental component Ψ_{π_0} corresponding to the link pattern π_0 that connects points $i \leftrightarrow 2n+1-i$, $i = 1, 2, \dots, n$, and then that the latter reads:

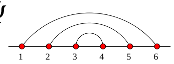
$$\Psi_{\pi_0} = \prod_{1 \le i < j \le n} (qz_i - q^{-1}z_j) \prod_{n+1 \le i < j \le 2n} (qz_i - q^{-1}z_j) \tag{3.5}$$

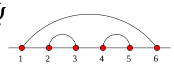
In [2], an expression for the sum rule $\sum_\pi \Psi_\pi$ was derived based on symmetry properties and recursion relations inherited from eq.(3.3). It was then realized first in the language of affine Temperley-Lieb algebra [4], and then in that of the qKZ equation [5], that a

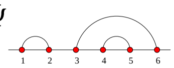
natural extension of this holds for generic q as well, at the expense of giving up the loop gas picture. Indeed, the system (3.1) is nothing but a particular case of qKZ equation for the value $q = -e^{i\pi/3}$. The generic q equation reads

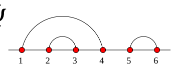
$$\begin{aligned} \tau_{i,i+1}\Psi(z_1,\ldots,z_{2n}) &= \check{R}_{i,i+1}(z_i,z_{i+1})\,\Psi(z_1,\ldots,z_{2n}), \quad i=1,2,\ldots 2n-1 \\ \sigma\,\Psi(z_2,\ldots,z_{2n},q^6 z_1) &= q^{3(n-1)}\,\Psi(z_1,\ldots,z_{2n}) \end{aligned} \tag{3.6}$$

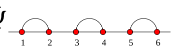
with $\check{R}_{i,i+1}$ and σ as before, but for q generic (note that the e_i are now generators of the Temperley-Lieb algebra with weight τ per loop, as opposed to 1 before). Remarkably the generic q minimal polynomial solution to these equations is essentially the same, namely that every component is obtained triangularly in terms of Ψ_{π_0} by successive actions with divided difference operators and multiplications by monomials, and that the expression for Ψ_{π_0} remains the same as in (3.5), but now with q generic. For illustration, in the case $n = 3$, there are 5 link patterns with $2n = 6$ boundary points (which we conveniently represent along a line rather than on a circle, as the cyclic invariance is now lost, and we find that

Ψ  $= a_{1,2}a_{1,3}a_{2,3}a_{4,5}a_{4,6}a_{5,6}$

Ψ  $= a_{1,2}a_{3,4}a_{5,6}(a_{1,3}a_{4,6}b_{5,2} + a_{2,4}a_{3,5}b_{6,1})$

Ψ  $= a_{2,3}a_{2,4}a_{3,4}a_{5,6}b_{5,1}b_{6,1}$ (3.7)

Ψ  $= a_{1,2}a_{3,4}a_{3,5}a_{4,5}b_{6,1}b_{6,2}$

Ψ  $= a_{2,3}a_{4,5}b_{6,1}(a_{2,4}b_{5,1}b_{6,3} + a_{1,2}a_{3,5}a_{4,6})$

Here we have introduced a convenient notation:

$$a_{i,j} = qz_i - q^{-1}z_j, \qquad \text{and} \qquad b_{i,j} = q^{-2}z_i - q^2 z_j \tag{3.8}$$

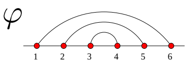
Note that these two definitions coincide at the RS point $q = -e^{i\pi/3}$.

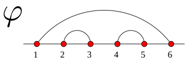
We may then consider the renormalized probabilities Ψ_π in the homogeneous limit where all the z_i's are taken to 1, while q remains fixed and generic. Actually, the correct choice of normalization in this case is really to consider $\varphi_\pi = \Psi_\pi/\Psi_{\pi_0}$, so that $\varphi_{\pi_0} = 1$. It turns out that all quantities φ_π are polynomials of $\tau = -q - q^{-1}$ with non-negative integer coefficients. Consequently, the sum rule

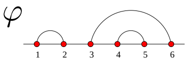
$$\Pi_{2n}(\tau) \equiv \sum_\pi \frac{\Psi_\pi(1,1,\ldots,1)}{\Psi_{\pi_0}(1,1,\ldots,1)} \tag{3.9}$$

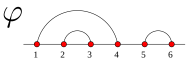
also produces polynomials of τ with non-negative integer coefficients.

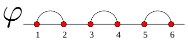
Returning to the example $n = 3$ of eq.(3.7), and noting that $b_{i,j} = \tau a_{i,j}$ in the homogeneous limit, we find, upon dividing the entries above by $\Psi_{\pi_0} = (q - q^{-1})^6$, that

φ  $= 1$

φ  $= 2\tau$

φ  $= \tau^2$ (3.10)

φ  $= \tau^2$

φ  $= \tau + \tau^3$

henceforth $\Pi_6(\tau) = 1 + 3\tau + 2\tau^2 + \tau^3$. Note that the powers of τ are simply equal to the numbers of b factors in each product of a's and b's forming the quantities Ψ_π. Note also the coincidence of $\Pi_6(\tau)$ with $P_6(\tau)$ of eq.(2.5).

4. The conjecture

4.1. The conjecture

We now have all the elements to formulate the following

Conjecture:

The polynomial $P_{2n}(\tau)$, equal to the generating function for TSSCPP of size $2n$, with a weight τ per vertical step (in NILP form), and the polynomial $\Pi_{2n}(\tau)$, equal to the homogeneous sum rule for the solution of the level-one $U_q(\mathfrak{sl}(2))$ qKZ equation at generic q, do coincide for all n, with $\tau = -q - q^{-1}$.

4.2. Checks and useful algorithms

A first non-trivial check of the above conjecture regards the values of the polynomials at $\tau = 2$. This corresponds to $q = -1$, where earlier findings [5] have shown that the quantities φ_π may be interpreted as the degrees of the components of the variety $M^2 = 0$, M an upper triangular complex matrix of size $2n \times 2n$. In particular, the total degree $d_{2n} = \Pi_{2n}(2)$ of this variety, should equal $P_{2n}(2)$. Computing this total degree numerically, we have found a nice formula for d_{2n} matching the results for sizes up to 12. It is again a

sum of determinants, which actually also counts yet another set of non-intersecting lattice paths. It reads:

$$d_{2n} = \sum_{\substack{1 \le r_1 < \ldots < r_{n-1} \\ r_i \le 2i}} \det_{1 \le i,j \le n-1} \binom{2i}{r_j} \tag{4.1}$$

and enumerates NILP that start at points $(2i, 0)$, end up at points (r_i, r_i), and take only vertical steps $(0, 1)$ and horizontal steps $(-1, 0)$, with $i = 1, 2, \ldots, n-1$.

To show that $d_{2n} = P_{2n}(2)$, let us now prove that the rectangular $n-1 \times 2n-2$ matrices B and A with entries $B_{i,r} = \binom{2i}{r}$ and $A_{i,r} = \binom{i}{r-i} 2^{2i-r}$ actually share the same minors of size $n-1 \times n-1$. To prove this, it is sufficient to find a square matrix Q of determinant 1, such that $B = QA$. It turns out that the lower triangular matrix with entries $Q_{k,i} = \binom{k}{i}$ does the job. Indeed, let us compute the following generating function for the entries of QA:

$$\begin{aligned} a_k(x) &\equiv \sum_{i,r \ge 1} x^r \binom{k}{i} \binom{i}{r-i} 2^{2i-r} \\ &= \sum_{i \ge 1} (2x)^i \binom{k}{i} \sum_{m \ge 0} \binom{i}{m} \left(\frac{x}{2}\right)^m \\ &= \sum_{i \ge 1} (2x + x^2)^i \binom{k}{i} = (1+x)^{2k} - 1 \end{aligned} \tag{4.2}$$

henceforth, picking the coefficient of x^r for $r \ge 1$, we end up with $B = QA$.

We have checked the general conjecture for arbitrary τ up to $n = 6$. The computation of $P_{2n}(\tau)$ is straightforward, thanks to the formula (2.11). On the other hand, that of the sum rule $\Pi_{2n}(\tau)$ is more subtle, as it involves constructing the solution of the qKZ equation explicitly, and only in the end taking the homogeneous limit. To do this, we have found a powerful algorithm, that might eventually help prove the RS conjecture. It is based on the following simple remark on the master equation

$$(q^{-1} z_{i+1} - q z_i) \partial_i \Psi_\pi = \sum_{\substack{\pi' \ne \pi \\ e_i \pi' = \pi}} \Psi_{\pi'}, \quad i = 1, 2, \ldots 2n-1 \tag{4.3}$$

when π is in the image of e_i, which allows to compute all Ψ_π's in a triangular way in terms of Ψ_{π_0}.

This goes as follows. We first represent the link patterns as a set of n non-intersecting semi-circles connecting $2n$ regularly spaces points along a line, labeled from 1 to $2n$, and

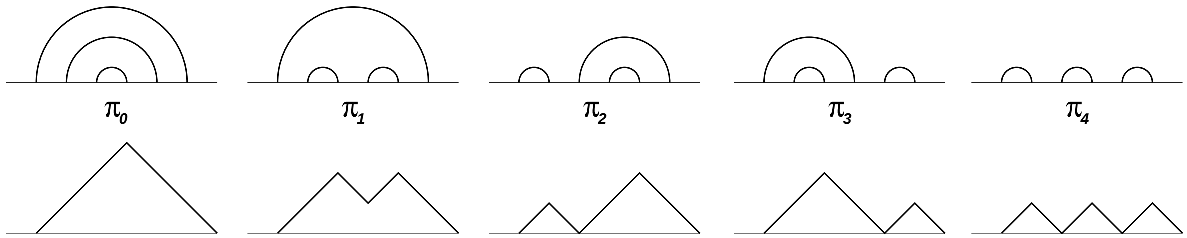


Fig. 5: The bijection between link patterns and Dyck paths in size $2n = 6$. The corresponding inclusion order on link patterns reads: $\pi_0 < \pi_1 < \pi_2, \pi_3 < \pi_4$.

then use a standard bijection to Dyck paths (see Fig.5 for the case $2n = 6$), namely paths of length $2n$ in the plane starting from the origin and taking only diagonal steps $(1,1)$ or $(1,-1)$, ending at $(2n,0)$, and visiting only points (x,y) with $y \geq 0$. The bijection is obtained as follows: visiting the points of the link pattern from left to right, we form a Dyck path by taking a i-th step $(1,1)$ if a semi-circle originates from the point i or $(1,-1)$ if a semi-circle terminates at point i. We may now define an order on link patterns $\pi < \pi'$ iff the Dyck path of π *contains* that of π', namely for each i-th visited point (x_i, y_i) and (x'_i, y'_i) of the respective paths, the difference $(y_i - y'_i) \geq 0$, and at least one of these quantities is non-zero. Now given a link pattern π in the image of e_i, its antecedents $\pi' \neq \pi$ under e_i are all such that $\pi' < \pi$, except for one, say π^*, such that $\pi^* > \pi$. In the Dyck path language, π^* is identical to π except that its steps i and $i+1$ are interchanged. Therefore, triangularly w.r.t. inclusion of Dyck paths, we may determine recursively

$$\Psi_{\pi^*} = (q^{-1} z_{i+1} - q z_i) \partial_i \Psi_\pi - \sum_{\substack{\pi' \neq \pi, \pi^* \\ e_i \pi' = \pi}} \Psi_{\pi'} \tag{4.4}$$

The main difficulty here is to actually evaluate the l.h.s. and to put it in a suitable form. We have found that each quantity Ψ_π may be written as a *sum of products* of $a_{i,j}$ and $b_{k,l}$'s of eq.(3.8), with $i < j$ and $k > l$, without any additional numerical factor (see eq.(3.7) for an example at $2n = 6$). The problem is that the expressions as sums are not unique, but their homogeneous values are, even at generic q. So the algorithm simply expresses the Ψ_π, order by order in the computation, as such sums of products. More precisely, we start from

$$\Psi_{\pi_0} = \prod_{1 \leq i < j \leq n} a_{i,j} \prod_{n+1 \leq i < j \leq 2n} a_{i,j} \tag{4.5}$$

and write eq.(4.3) for $i = n$ (which is the only place where the link pattern π_0 has a chord linking points i and $i+1$, hence is the image under e_i of some other link pattern). Acting with ∂_i on any symmetric polynomial of z_i, z_{i+1} gives zero by definition, hence ∂_n has a non-trivial action only on the part of Ψ_{π_0} that is not symmetric in z_n, z_{n+1}, namely on $\prod_{i=1}^{n-1} a_{i,n} \prod_{j=n+2}^{2n} a_{n+1,j}$. To proceed, we now need two lemmas on the action of ∂_i on monomials.

Lemma 1:

The divided difference operator obeys a modified Leibnitz rule:

$$\partial_i(fg) = \partial_i(f)g + t_{i,i+1}(f)\partial_i(g) \tag{4.6}$$

as a straightforward consequence of the definition (3.4).

Lemma 2: The operator $T_i \equiv (q^{-1}z_{i+1} - qz_i)\partial_i$ acts as follows on products of two monomials:

$$\begin{aligned} T_i(a_{n,i}\, a_{i+1,p}) &= a_{i,i+1}\, b_{p,n}, \quad \text{for } n < i < i+1 < p \\ T_i(a_{p,i}\, b_{i+1,n}) &= a_{i,i+1}\, a_{n,p}, \quad \text{for } n < p < i < i+1 \\ T_i(b_{n,i}\, a_{i+1,p}) &= a_{i,i+1}\, a_{n,p}, \quad \text{for } i < i+1 < n < p \end{aligned} \tag{4.7}$$

We may now explain the algorithm: starting from a Ψ_π already expressed as a sum of products of a's and b's, apply the operator T_i on each such product. First group the terms of the product that are non-symmetric in z_i, z_{i+1} into pairs of the form aa or ab like in the l.h.s. of eq.(4.7)(it turns out that this can always be done, possibly in several ways). Then use the modified Leibnitz rule (4.6) to express the action of T_i as a sum of terms in which T_i only acts on one pair. Use (4.7) to express this action. We are left with a sum of products of a's and b's. The only difficulty is to make sure indices in the pairs are ordered as in (4.7), but then the action is straightforward. There are unfortunately many choices of pairs, some of which may lead to dead ends, where no use of (4.7) may be allowed, hence the algorithm must explore many possibilities before converging.

A time-saving remark however is that we have not made use of the quasi-cyclicity relation

$$q^{-3(n-1)}\Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1) = \Psi_{\sigma(\pi)}(z_1, \ldots, z_{2n}) \tag{4.8}$$

which is actually granted once we pick Ψ_{π_0} as in eq.(3.5). It is easy to see that this relation may be applied to any product of a and b monomials, in which the index $2n$ appears exactly $n-1$ times (a fact easily checked for the products of a's and b's we generate), and that under the action of σ we have $a_{i,j} \to a_{i+1,j+1}$ unless $j = 2n$, in which case $a_{i,2n} \to b_{i+1,1}$, and

$b_{i,j} \to b_{i+1,j+1}$, unless $j = 2n$, in which case $b_{i,2n} \to a_{1,i+1}$. So the quasicyclic invariance condition (4.8) allows to generate all the rotated versions $\Psi_{\sigma^r(\pi)}$ as sums of products of a's and b's out of an expression for Ψ_π as a sum of products of a's and b's. Combining this with the main algorithm, we have been able to generate solutions of qKZ up to size $2n = 12$, and found perfect agreement with the conjecture of Sect. 4.1.

5. Consequences and conclusion

In this note we have presented a new conjecture relating TSSCPP in the form of NILP and ASM or generalized sum rules for the qKZ equation. The algorithm developed in Sect. 4.2 above, might help find a bijection between ASM and TSSCPP. Indeed, comparing the generic q situation with that of the RS point, where there is no more distinction between a's and b's, and assuming the RS conjecture to be true, there are exactly as many products of a's and b's forming Ψ_π at generic q as there are FPL configurations with the link pattern π. So these expressions (unfortunately not unique at this point, but one may hope for the existence of a canonical decomposition) of Ψ_π as sums of products of a's and b's suggest to attach each such product to each FPL configuration with link pattern π on one hand, and to some TSSCPP with as many vertical steps as there are b's in the product, on the other hand. Crossing this information with that of refined ASM and TSSCPP of Ref.[11], this might eventually lead to a bijection between ASM and TSSCPP, and perhaps to a proof of the RS conjecture.

Another interesting consequence of the conjecture regards the point $q = -1$, namely $\tau = 2$. As mentioned in Sect. 4.2, the quantity $\Pi_{2n}(2) = d_{2n}$ counts the total degree of the variety $M^2 = 0$ for upper triangular complex matrices M. $P_{2n}(2)$, when expressed as a sum over NILP yields via the above conjecture a formula for the total degree d_{2n} as a sum of powers of 2 (each a contributes 1 and each b contributes 2 in the homogeneous case $z_i = 1$ followed by the $q \to -1$ limit of φ_π), each term in the sum corresponding to a TSSCPP. The TSSCPP therefore play the role of "pipe dreams" [12] for the degree counting, and suggest that the variety $M^2 = 0$ may be decomposed into complete intersections of linear hyperplanes and quadratic (degree 2) varieties, and that this decomposition involves exactly as many terms (components) as the number of ASM or TSSCPP. Such an algebro-geometric interpretation of ASM or TSSCPP numbers would be quite nice.

Another point of interest is $q = e^{i\pi/3}$, i.e. $\tau = -1$, where our conjectured relation between $P_{4n+2}(-1)$ and the square of the number of VSASM of size $(2n+1) \times (2n+1)$,

also yields via the main conjecture of this paper a relation between the numbers $F_n(\pi)$ of FPL configurations of size $n \times n$ with link pattern π, and the number $A_V(n)$. Indeed, the quantities Ψ_π enjoy some parity property that all products of a's and b's in their decomposition have numbers of b of the same parity. Hence in the homogeneous case, the corresponding polynomial $\varphi_\pi(1,1,...,1)$ of τ has a fixed parity (see eq.(3.10) for the example $2n=6$). It appears that this parity is reversed under the action of e_i (when it is non-trivial, i.e. when no chord connects i and $i+1$ in π), namely parity$(\varphi_{e_i\pi}) = -$parity(φ_π): this is a direct consequence of the action (4.7), which adds up or removes a b, therefore reverses the parity of the homogeneous quantities. With the normalization $\varphi_{\pi_0}=1$, this fixes all parities of the φ_π, say ϵ_π. So when $\tau=-1$, we get a new alternating sum rule

$$\sum_\pi \epsilon_\pi \, F_n(\pi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} A_V(n)^2 & \text{if } n \text{ is odd} \end{cases} \tag{5.1}$$

while $\sum_\pi F_n(\pi) = A_n$.

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